Synopsis

1. **NAME OF SCHOLAR**: Rashmi Goyal
2. **REGISTRATION NUMBER**: VGU/2015/RES/SCH/MATH/1002
3. **TITLE OF RESEARCH**: A study of Special Functions, Derivatives and Integrals of Arbitrary Order and Fractional Differential Equations with Applications.
4. **DEPARTMENT**: Faculty of Basic and Applied Sciences, Department of Mathematics, Vivekananda Global University, Jaipur.
5. **OBJECTIVE AND SCOPE OF RESEARCH INVESTIGATION**

Study of **Special Functions** is of great importance due to their applications in solving various problems arising in physical, biological and engineering sciences. Special functions are a class of mathematical functions that arise in the solution of various classical problems of physics. These problems generally involve the flow of electromagnetic, acoustic, or thermal energy. Recently a number of researchers have shown that several special functions like Mittag- Leffler's function, the function E_t, C_t, S_t introduced by Miller and Ross, the F-function, the R-function provide direct solution and important understanding of the fractional order differential and integral equations and the related boundary value problems occurring in such areas as electromagnetic theory, probability theory, electrical networks, micro elasticity, dissemination of atmospheric pollutants. These functions as well as number of other useful special functions follow as simple special cases of a function introduced by Charles Fox popularly known as H-function.The H-function is further generalized by Inayat Hussain (1987) as I-function. This function is studied in detail by Buschman and Srivastava (1990), Saxena (1998) and several others. Theory of functions involving two and several variables was developed by a number of research workers namely Appell, Humbert, Horn, Exton, Pathan, Agrawal, Lauricella, Srivastava, Daoust and several others.

In the present work we propose to establish some new results for various special functions and polynomials. We also propose to define a new function in two or more variables as a generalization of some existing function of one variable.

Study of **Integrals and Derivative to an Arbitrary Order** i.e. **Fractional Calculus** is a significant topic in mathematical analysis as a result of its increasing range of applications. Fractional calculus is a field of mathematical study that grows out of the traditional definition of the integer order calculus of derivatives and integrals. It provide several tools for solving differential and integral equations of fractional order. The Fractional Calculus is as old as classical calculus but has gained importance during the past three decades only, due mainly to its applications in numerous diverse fields of science and engineering. Some of the areas of applications of Fractional Calculus include fluid flow, rheology, diffusive transport, electrical network, probability and statistics, control theory of dynamical systems, visco elasticity, chemical physics, optics, signal processing and several others. A growing number of works in science and engineering deal with dynamical systems described by **Fractional Order Equations** that involve

derivatives and integrals of non-integer order as these describe the memory and hereditary properties of different substances. Future scope in Fractional Calculus is important developments in classical analysis, linear and nonlinear functional analysis, ordinary and partial differential equations, optimization theory, control theory and several others.

In fractional calculus, we propose to study and develop some new fractional integral/differential operators. We also propose to find the solutions of various fractional differential equations using the technique of integral transform, Adomian decomposition method or homotopy analysis method.

Statistical distributions involving special functions have been studied from time to time by several research workers due to their applications in various fields. In particular the distributions involving Bessel function have found applications in a variety of areas that range from image and speech recognition and ocean engineering to finance. Such distributions are also useful in the treatment of various military operations research problems related to radar discrimination. Distributions involving more general functions such as generalized hyper geometric function and Fox H-function have also been defined by various authors. The problem of deriving the distribution of algebraic functions such as sum, difference, product, ratio and linear combination of random variables, where the individual random variable follows a particular probability density function, occurs in wide variety of areas. For example, sum of independent gamma random variables have application in problems of queuing theory such as determination of the total excess water flow in a dam. A simple but practical problem requiring the product of independent random variables concerns signal amplification. If n amplifiers are connected in series and if X_i denotes the amplification of the i^{th} amplifier, the analysis of the total amplification $Y = X_1 X_2 \ldots X_n$ is basically a problem in the analysis of product of independent random variables. An important example of ratio of random variables is the stress-strength model in the context of reliability.

In statistical distribution, we propose to study the probability distribution function of some of three triangular random variables. We also propose to define new probability distribution function by combining to known distribution function and study them.

6. PROPOSED METHODOLOGY:

There are several methods to solve Fractional differential equations. Some methods have been developed to obtain the exact and approximate analytic solutions of Fractional differential equations. Some of these methods use integral transform in order to reduce equations into simpler equation, and some other methods give the solutions in a series form which converges to the exact solution. We can solve Fractional differential equations by using these types of methods.

Some more methods used are Adomian decomposition method, Differential transform method (by Zhou and it constructs an analytical solution in the form of a polynomial) and/ Matrix method/homotopy analysis method.

There are several approaches to the generalization of the notion of differentiation to fractional orders, for example, Riemann-Liouville, Grünwald-Letnikov, Caputo and generalized function approach.The Riemann- Liouville fractional derivative is mostly used but this approach is not very suitable for real world physical problems. Caputo introduced alternative definition which has the advantage of defining integer order initial condition for fractional order differential equations unlike the Riemann-Lioville approach which derives its definition from repeated integration.The Grünwald -Letnikov formulation approaches the problem from derivative side. This approach is mostly used in numerical algorithms. Grünwald-Letnikov derivative is a basic extension of the derivative in fractional calculus. In our work we shall use some such suitable approaches. We shall also use the softwares MATLAB/MAPLE/MATHEMATICA in solving equations and plotting graphs.

Limitations:

The advantages of Fractional Calculus are good data fitting, non-locality description and easy to use but disadvantages include physical interpretation, parameters determination and three dimensional analysis. Actually rigorous mathematical theory has been developed. The geometrical interpretation or physical meaning exists but it is not as straight forward as for the integer-order derivative. Due to some fractional powers, MATLAB computations are not so easy and may create some problems to solve some difficult equations. Fractional differential equations are indeed concepts of a higher complexity by nature but once mastered the idea, it can be solved.

7. IMPORTANCE OF PROPOSED RESEARCH/EXPECTED OUTCOMES:

Mathematical modeling of real life problems usually results in Fractional differential equations and various other problems involving special functions of mathematical physics as well as their extensions and generalization in one or variables .there are fluid dynamics, quantum mechanics, electricity, ecological systems and many other modes are controlled within their domain of validity by fractional order PDEs.

In the recent years, fractional calculus has played a very important role in various fields. Based on the wide applications in engineering and sciences such as physics, mechanics, chemistry, biology, etc.

There are several applications of fractional differential equations.

- Viscoelastic materials
- Polymeric materials
- Fluid flow
- Dynamical process with self similar structures
- Optics

- Geology
- Biosciences
- Medicine
- Dynamics of Earthquakes
- wave propagation
- random walk theory
- signal analysis
- Control theory in engineering

Recently the importance of fractional differential equations has been studied-The following areas are-

The fractional diffusion equation has been explicitly introduced in physics by nigmatullin to describe diffusion in media with fractal geometry (special types of porous media). Fractional wave equation governs the propagation of me- chanical diffusive waves in viscoelastic media which exhibit a power-law creep.

The space-Time fractional telegraph equations is used in: signal analysis for transmission and propagation of electrical signals, modeling reaction diffusion in various branches of engineering sciences in the propagation of pressure waves in the study of pulsatile blood flow in arteries and in one-dimensional random motion of bugs along a hedge.
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Applications in Medicine: Medical imaging is often design being a sub discipline of biomedical engineering like radiology, ultra sonography, etc. Actually Fractional differentiation plays a role in image and signal processing and algorithm is formed for image manipulation.

Applications in analysis of Dynamics of Earthquakes: The Fractional calculus is used extensively in analysis of Earthquakes. Some papers analyze earthquake data in the perspective of dynamical system and fractional calculus. The new stand point uses multi dimensional scaling (MDS) as a powerful and visualization tool. MDS is a technique that produces spatial or geometric representation of a complex objects. MDS constitutes a valid alternative to classic visualization tools for understanding the global behavior of earthquakes

Therefore, it becomes increasingly important to be familiar with all the traditional and recently developed methods for solving fractional order equations and the implementations of these methods.

8. REVIEW OF LITERATURE

The applications of fractional calculus have been studied by many researchers. So name a few, the nonlinear oscillation of earthquakes given by He (1998), fluid-dynamic traffic model given by He (1999), to model frequency dependent damping behavior of many visco elastic materials given by Bagley and Torvik (1983, 1985), continuum and statistical mechanics given by Mainardi (1997), colored noise given by Mandelbrot (1967), solid mechanics given by Rossikhin and Shitikova (1997), economics given by Baillie (1996), bioengineering given by Magin (2004), anomalous transport given by Metzler and Klafter (2004), and dynamics of interfaces between nano particles and substrates given by Chow (2005). In mechanics fractional-order derivatives have been successfully used to model damping forces with memory effect or to describe state feedback controllers as shown by Bagley and Torvik (1983, 1984) and Tenreiro Machado (2009).It is found by Wang and Hu (2009) that in fractional-order vibration systems of single degree of freedom, the term of fractional-order derivative whose order is between 0 and 2 acts always as damping force. In addition, almost all systems containing internal damping are not suitable to be described properly by the classical methods, but the fractional calculus represents one of the promising tools to incorporate in a single theory both conservative and non conservative phenomena shown by Baleanu (2009). It is a well-recognized belief that fractional calculus leads to better results than classical calculus as shown in Podlubny (1999. The generalization of the fractional diffusion-advection equation in space-time exhibits anomalous behavior. Initial-Boundary Value Problem for Fractional Partial Differential Equations of Higher Order by Amanov, Djumaklych; Ashyralyev, Allaberen (2012).In this paper, the initial-boundary value problem for partial differential equations of higher-order involving the Caputo fractional derivative is studied. Theorems on existence and uniqueness of a solution and its continuous dependence on the initial data and on the right-hand side of the equation are established.). Convergence of Variational Iteration Method for Solving Singular Partial Differential Equations of Fractional Order **by** Elbeleze, Asma Ali; Kılıçman, Adem; Taib, Bachok M.(2014).In this paper singular partial differential equations of fractional order (FSPDEs). The variational iteration method (VIM) is applied to obtain approximate solutions of this type of equations. Convergence analysis of the VIM is discussed. This analysis is used to estimate the maximum absolute truncated error of the series solution. A comparison between the results of VIM solutions and exact solution is given. The fractional derivatives are described in Caputo sense. Space-Time Fractional Diffusion-Advection Equation with Caputo Derivative by Gómez Aguilar, José Francisco; Hernández, Margarita Miranda(2014).In this paper, an alternative construction for the space-time fractional diffusion-advection equation for the sedimentation phenomena is presented.

9. RESEARCH GAP

Special functions such as Legendre and Bessel were primarily defined as solution of differential equations occurring frequently in physics but as the study grew up and several of special functions were defined. We see that most of them don't satisfy any differential equations. This is a big gap in this field of research which one can try to fill.

Factorial function is a generalization of gamma function but there is no existence of any equation of gamma function this is also a gap in research

The problems formulated in the fractional calculus framework often require numerical fractional integration/differentiation of large data sets. Several existing fractional control toolboxes are capable of performing fractional calculus operations, however, none of them can efficiently perform numerical integration and differentiation on multiple large data sequences.

Actually we have to define some parameters in MATLAB, but they may create some problems, due to its complex calculations. For example, the Grunwald- Letnikov definition specifies what is essentially an infinite sum, what number of these terms must be computed and summed for an accurate result to be achieved. Because of the speed of modern computing equipment and the relative simplicity of the step by step calculation (except perhaps the calculation of the factorial), one might suppose that anywhere from 10000 to 1000000 steps would provide excellent accuracy without a significant penalty in computation time. In theory, this would be a correct assumption, as even the calculation of Gamma is done approximately in most mathematical software packages including MATLAB. However the limiting factor to the accuracy of this numerical solver is not the speed of the computer. Rather it is the capacity of the computer to accurately store and move the numbers that are being supplied it. But based on Fourier series and Taylor series theory computations of fractional derivatives by mathematical software such as MATLAB, we can efficiently compute fractional derivatives. Several existing fractional control toolboxes are capable of performing fractional calculus operations; however, none of them can efficiently perform numerical integration on multiple large data sequences.

10. CHAPTER WISE DETAILS:

We propose to form seven chapters.

1. Introduction of the research topic.

2-3. On Special Functions.

4-5. On Fractional Calculus.

6-7. On statistical distributions.

11. EXPECTED DURATION: 2- 3Years

12. FACILITIES:

The online/offline library facility for literature review will be required .The subscriptions of reputed online/offline journals of mathematics are required to study recent developments of the subject. The internet facilities must be provided by the Vivekananda Global University for download of full length papers which will help in doing further research. A proper sitting area with computer facility for sitting will be required in the university to carry out the research work effectively and efficiently.

13. BIBLIOGRAPHY

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Date: 30.11.2016